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Positive Definite Spectral Estimate and Stable Correlation Recursion for Multivariate Linear Predictive Spectral Analysis

Albert H. Nuttall
Special Projects Department

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PREFACE

This research was conducted under NUSC Project No. A-752-05, "Applications of Statistical Communication Theory to Acoustic Signal Processing," Principal Investigator — Dr. A. H. Nuttall (Code 313), and Navy Subproject and Task No. ZR-000-01, Program Manager — J. H. Probus (NAVMAT, MAT-035).

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The questions regarding a positive definite spectral estimate and a stable correlation recursion ~~(raised in NUSC Technical Report 5501)~~, are answered in the affirmative for the particular choice of weighting recommended in the above reference. A modified and updated FORTRAN program for multivariate spectral analysis, which incorporates calculation of the correlation matrices via recursion, and the aliased correlation matrices via a fast Fourier transform (FFT), are included.

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LIST OF SYMBOLS*

M_{p-1}	Auxiliary matrix
$R_m^{(p)}$	m-th order correlation matrix
$\mathcal{R}_m^{(p)}$	Block Toeplitz matrix
$Q_m^{(p)}$	Auxiliary block matrix
w_k, $\tilde{w}_k$	Trapezoidal weights
$\hat{R}_m$	Aliased correlation matrix
G_k	$G\left(\frac{k}{N_F\Delta}\right)$
$G_k^{(\ell j)}$	Element ℓ,j of G_k
u_k	Auxiliary scalar sequence
FFT	Fast Fourier transform

*This list of symbols is supplementary to that in an earlier report,[1] to which this report is a sequel.

POSITIVE DEFINITE SPECTRAL ESTIMATE AND STABLE CORRELATION RECURSION FOR MULTIVARIATE LINEAR PREDICTIVE SPECTRAL ANALYSIS

INTRODUCTION

A generalization of Burg's algorithm for spectral analysis to the multivariate case was the subject of an earlier report.[1] All the desirable properties of the univariate case were shown to hold true, except that it was not proven that the residual matrix was positive definite, nor that the correlation recursion was stable. Both of these assumptions can be affirmed by drawing on the results in Strand[2] and Burg.[3]

In addition to affirming these two assumptions, this report contains a modified and updated FORTRAN program that supersedes the program previously reported.[1] The modified program incorporates some more-explanatory format statements, the calculation of the (normalized) correlation matrices via recursion, and the aliased (normalized) correlation matrices by means of a Fast Fourier Transform (FFT).

This report is a sequel to an earlier report.[1] In order to eliminate duplication, that report is referenced for background information, a list of symbols used, and processing technique. We shall draw freely on that report; for example, equation (5) of the earlier report will be denoted by (5).[1]

POSITIVE DEFINITE RESIDUAL MATRIX

The (p-1)-th order forward residual matrix, U_{p-1}, was defined in equation (95).[1] We wish to show that U_p is positive definite; the following proof is based on reference 2, equations (3.25-3.32).

From equation (H-5),[1] we have, using the Hermitian property of U_p and V_p,

$$U_p = U_{p-1} - A_p^{(p)} V_{p-1} A_p^{(p)H} ; \tag{1}$$

and from equation (137),[1] eliminating $B_p^{(p)H}$,

$$A_p^{(p)} V_{p-1} = U_{p-1} S_{p-1}^{(yy)^{-1}} \left(2 S_{p-1}^{(yz)} - A_p^{(p)} S_{p-1}^{(zz)} \right). \tag{2}$$

Notice that we have made specific use of the inverse weighting in equation (136).[1] Substituting equation (2) into equation (1), we find

$$U_p = U_{p-1} - U_{p-1} S_{p-1}^{(yy)^{-1}} \left(2S_{p-1}^{(yz)} - A_p^{(p)} S_{p-1}^{(zz)} \right) A_p^{(p)H} \quad ; \tag{3}$$

therefore,

$$S_{p-1}^{(yy)} U_{p-1}^{-1} U_p = S_{p-1}^{(yy)} - 2S_{p-1}^{(yz)} A_p^{(p)H} + A_p^{(p)} S_{p-1}^{(zz)} A_p^{(p)H} \ . \tag{4}$$

Taking the conjugate transpose of both sides of equation (4) and using equations (106)[1] and (114)[1] yields

$$U_p U_{p-1}^{-1} S_{p-1}^{(yy)} = S_{p-1}^{(yy)} - 2A_p^{(p)} S_{p-1}^{(yz)^H} + A_p^{(p)} S_{p-1}^{(zz)} A_p^{(p)H} \ . \tag{5}$$

Adding equations (4) and (5) together and multiplying by -1, there follows

$$\begin{aligned} &\left(-S_{p-1}^{(yy)} U_{p-1}^{-1}\right) U_p + U_p \left(-U_{p-1}^{-1} S_{p-1}^{(yy)}\right) \\ &= -2\left[S_{p-1}^{(yy)} - A_p^{(p)} S_{p-1}^{(yz)^H} - S_{p-1}^{(yz)} A_p^{(p)H} + A_p^{(p)} S_{p-1}^{(zz)} A_p^{(p)H} \right] = -2E_p \ ; \end{aligned} \tag{6}$$

the last identity was derived from equation (113).[1]

Define

$$M_{p-1} = -U_{p-1}^{-1} S_{p-1}^{(yy)} \ . \tag{7}$$

Then equation (6) becomes simply

$$M_{p-1}^H U_p + U_p M_{p-1} = -2E_p \ . \tag{8}$$

Now, E_p is Hermitian and positive definite* (see equation (112)[1]); also, $S_{p-1}^{(yy)}$ is Hermitian and positive definite (see equation (114A)[1]). We assume that U_{p-1} is positive definite. Then, U_{p-1}^{-1} is positive definite, and so $U_{p-1}^{-1}\ S_{p-1}^{(yy)}$ must have all its eigenvalues positive

---

*All of the positive definite statements should be qualified with the proviso "with probability 1."

(see appendix A). As a result, M_{p-1} has all its eigenvalues negative, making it a stable matrix (reference 4, page 270). Therefore, the solution of equation (8) exists and is unique (reference 5, equation 3).

According to reference 4, page 278, problem 3, there exists a positive definite solution of equation (8) for U_p. Therefore, there is a unique positive definite solution of equation (8) for U_p. Since

$$U_0 = R_0 = \frac{1}{N} \sum_{k=1}^{N} X_k X_k^H \tag{9}$$

(from equations (95)[1] and (82)[1]) is positive definite, the assumption above, that U_{p-1} is positive definite, can be justified by induction.

In summary, the residual matrix U_p, calculated by means of equation (105)[1] or (181),[1] is positive definite. The quantity V_p is also positive definite; the equation analogous to equation (6) is

$$\left(-S_{p-1}^{(22)} V_{p-1}^{-1}\right) V_p + V_p \left(-V_{p-1}^{-1} S_{p-1}^{(22)}\right) = -2 F_p \, , \tag{10}$$

and all the comments above apply directly. It is worth repeating that the positive definite conclusion on U_p and V_p holds for the specific inverse weighting indicated in equation (136)[1]; whether it also holds for other weightings is unknown.

STABLE CORRELATION RECURSION

The correlation recursion is given in equation (164)[1] according to

$$R_m^{(p)} = \sum_{n=1}^{p} A_n^{(p)} R_{m-n}^{(p)} \, , \quad p+1 \le m,$$

$$R_m^{(p)} = R_{-m}^{(p)H} \, , \quad m < 0, \tag{11}$$

where superscript p has been added to the correlation matrices to indicate specifically their dependence on the p-th order predictive filter; and starting values have been defined, as in equation (D-3),[1] namely,

$$R_m^{(p)} = R_m \,, \quad |m| \le p. \tag{12}$$

The latter quantities in equation (12) are, according to equation (78A),[1] solutions of

$$R_m = \sum_{n=1}^{p} A_n^{(p)} R_{m-n} \,, \quad 1 \le m \le p. \tag{13}$$

Combining equations (11) through (13), we have

$$R_m^{(p)} = \sum_{n=1}^{p} A_n^{(p)} R_{m-n}^{(p)} \,, \quad 1 \le m. \tag{14}$$

We will show that recursion (11) is stable; that is, we will show that (the elements of) matrix $R_m^{(p)}$ does not tend to infinity as m tends to infinity, with p fixed. The proof is an extension of reference 3, section III.C.2 (which was for known correlation), to fit the unknown correlation case.

We have, from equations (82)[1] and (80A),[1] respectively,

$$R_0 = \frac{1}{N} \sum_{k=1}^{N} X_k X_k^H \,,$$

$$R_p = \sum_{n=1}^{p} A_n^{(p)} R_{p-n} \quad \text{for } p = 1, 2, \ldots \tag{15}$$

For a given value of p, define the (m + 1) × (m + 1) block Toeplitz matrix

$$\mathcal{R}_m^{(p)} \equiv \begin{bmatrix} R_0^{(p)} & R_1^{(p)} & \cdots & R_m^{(p)} \\ R_{-1}^{(p)} & R_0^{(p)} & & \\ \vdots & & \ddots & \\ R_{-m}^{(p)} & & & R_0^{(p)} \end{bmatrix} \tag{16}$$

If $m \leq p$, the entries in equation (16) are according to equation (12), whereas if $m > p$, the entries are those generated by equation (11). It follows immediately, from equations (16) and (12), that

$$\mathcal{R}_m^{(p)} = \mathcal{R}_m^{(m)} \text{ if } m \leq p. \tag{17}$$

The s,t-th block of $\mathcal{R}_m^{(p)}$ in equation (16) is

$$\left\{\mathcal{R}_m^{(p)}\right\}_{st} = R_{t-s}^{(p)} \text{ for } 0 \leq s,t \leq m. \tag{18}$$

Also, define a $(m + 1) \times (m + 1)$ block matrix,

$$Q_m^{(p)} \equiv \begin{bmatrix} I & 0 & 0 & \cdots & & 0 \\ -A_1^{(p)H} & I & 0 & \cdots & & \\ -A_2^{(p)H} & 0 & I & \ddots & & \\ \vdots & 0 & 0 & & & \\ \vdots & \vdots & \vdots & & & \\ -A_p^{(p)H} & & & & & \\ 0 & & & & & \\ 0 & & & \ddots & \vdots & \\ & & & & I & 0 \\ \vdots & & & \cdots & 0 & I \end{bmatrix}, \tag{19}$$

where we require $m \geq p \geq 1$ for this definition. Then, using the notation established in equation (18),

$$\left\{Q_m^{(p)}\right\}_{tu} = \delta_{tu} I - \delta_{u0} \tilde{A}_t^{(p)H} \text{ for } 0 \leq t,u \leq m, \tag{20}$$

where

$$\tilde{A}_t^{(p)} \equiv \left\{ \begin{array}{l} A_t^{(p)}, \ 1 \le t \le p \\ 0, \text{ otherwise} \end{array} \right\}. \tag{21}$$

Also,

$$\left\{ Q_m^{(p)^H} \right\}_{rs} = \delta_{rs}\, I - \delta_{r0}\, \tilde{A}_s^{(p)} \quad \text{for } 0 \le r, s \le m. \tag{22}$$

Then, the r,u-th block of the product $Q_m^{(p)^H} \mathcal{R}_m^{(p)} Q_m^{(p)}$ is

$$\left\{ Q_m^{(p)^H} \mathcal{R}_m^{(p)} Q_m^{(p)} \right\}_{ru} = \sum_{s,t=0}^{m} \left\{ Q_m^{(p)^H} \right\}_{rs} \left\{ \mathcal{R}_m^{(p)} \right\}_{st} \left\{ Q_m^{(p)} \right\}_{tu}$$

$$= \sum_{s,t=0}^{m} \left[\delta_{rs} I - \delta_{r0} \tilde{A}_s^{(p)} \right] R_{t-s}^{(p)} \left[\delta_{tu} I - \delta_{u0} \tilde{A}_t^{(p)^H} \right]$$

$$= \sum_{s,t=0}^{m} \Big[\delta_{rs} \delta_{tu} R_{t-s}^{(p)} - \delta_{r0} \delta_{tu} \tilde{A}_s^{(p)} R_{t-s}^{(p)} - \delta_{rs} \delta_{u0} R_{t-s}^{(p)} \tilde{A}_t^{(p)^H} + \delta_{r0} \delta_{u0} \tilde{A}_s^{(p)} R_{t-s}^{(p)} \tilde{A}_t^{(p)^H} \Big]$$

$$= R_{u-r}^{(p)} - \delta_{r0} \sum_{s=0}^{m} \tilde{A}_s^{(p)} R_{u-s}^{(p)} - \delta_{u0} \sum_{t=0}^{m} R_{t-r}^{(p)} \tilde{A}_t^{(p)^H} + \delta_{r0} \delta_{u0} \sum_{s,t=0}^{m} \tilde{A}_s^{(p)} R_{t-s}^{(p)} \tilde{A}_t^{(p)^H} \tag{23}$$

$$= R_{u-r}^{(p)} - \delta_{r0} \sum_{s=1}^{p} A_s^{(p)} R_{u-s}^{(p)} - \delta_{u0} \sum_{t=1}^{p} R_{t-r}^{(p)} A_t^{(p)^H} + \delta_{r0} \delta_{u0} \sum_{s,t=1}^{p} A_s^{(p)} R_{t-s}^{(p)} A_t^{(p)^H}$$

In the last line, above, we have used equation (21) to simplify equation (23).

At this point, we consider four subcases:

(a) for $1 \le r, u \le m$, equation (23) reduces to $R_{u-r}^{(p)}$;

(b) for $r = 0$, $u = 0$, equation (23) becomes

$$R_0^{(p)} - \sum_{s=1}^{p} A_s^{(p)} R_{-s}^{(p)} - \sum_{t=1}^{p} R_t^{(p)} A_t^{(p)^H} + \sum_{s,t=1}^{p} A_s^{(p)} R_{t-s}^{(p)} A_t^{(p)^H} \; ; \tag{24}$$

but, by use of equation (14), the sum on s in the last term of equation (24) is $R_t^{(p)}$, in which case the last two terms of equation (24) cancel. We are left with

$$R_0^{(p)} - \sum_{s=1}^{p} A_s^{(p)} R_{-s}^{(p)} = - \sum_{s=0}^{p} A_s^{(p)} R_{-s} = U_p \, , \tag{25}$$

using equations (12) and (95)[1];

(c) for r = 0, $1 \leq u \leq m$, equation (23) yields

$$R_u^{(p)} - \sum_{s=1}^{p} A_s^{(p)} R_{u-s}^{(p)} = 0, \tag{26}$$

using equation (14); and

(d) for u = 0, $1 \leq r \leq m$, equation (23) yields

$$R_{-r}^{(p)} - \sum_{t=1}^{p} R_{t-r}^{(p)} A_t^{(p)^H} = 0, \tag{27}$$

since this is the conjugate transpose of equation (26). Therefore, we have

$$Q_m^{(p)^H} \mathcal{R}_m^{(p)} Q_m^{(p)} = \begin{bmatrix} U_p & 0 & 0 & \cdots & 0 \\ 0 & R_0^{(p)} & R_1^{(p)} & & R_{m-1}^{(p)} \\ 0 & R_{-1}^{(p)} & R_0^{(p)} & \ddots & \\ \vdots & & & & \\ \vdots & & & & \\ 0 & R_{1-m}^{(p)} & & & R_0^{(p)} \end{bmatrix} = \left[\begin{array}{c|ccc} U_p & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \mathcal{R}_{m-1}^{(p)} & \\ 0 & & & \end{array} \right] . \tag{28}$$

This relation holds for $m \geq p \geq 1$, as noted under equation (19) (some relations for determinants are noted in appendix B).

Now, let $\{\mathcal{V}_k\}$ be arbitrary nonzero complex $M \times 1$ column matrices. Then, using equation (28),

$$\left[\mathcal{V}_0^H \cdots \mathcal{V}_m^H\right] Q_m^{(p)^H} \mathcal{R}_m^{(p)} Q_m^{(p)} \begin{bmatrix} \mathcal{V}_0 \\ \vdots \\ \mathcal{V}_m \end{bmatrix} = \mathcal{V}_0^H U_p \mathcal{V}_0 + \left[\mathcal{V}_1^H \cdots \mathcal{V}_m^H\right] \mathcal{R}_{m-1}^{(p)} \begin{bmatrix} \mathcal{V}_1 \\ \vdots \\ \mathcal{V}_m \end{bmatrix} . \tag{29}$$

We recall that U_p is positive definite, by the previous section. Therefore, if $\mathcal{R}_{m-1}^{(p)}$ is positive definite, then $Q_m^{(p)^H}$ $\mathcal{R}_m^{(p)}$ $Q_m^{(p)}$ is positive definite, which, in turn, implies that $\mathcal{R}_m^{(p)}$ is positive definite. That is, for $m \geq p \geq 1$,

$$\text{if } \mathcal{R}_{m-1}^{(p)} \text{ is positive definite, then } \mathcal{R}_m^{(p)} \text{ is positive definite.} \tag{30}$$

In particular, letting $m = p$, we see that if $\mathcal{R}_{p-1}^{(p)}$ is positive definite, then $\mathcal{R}_p^{(p)}$ is positive definite. But $\mathcal{R}_{p-1}^{(p)} = \mathcal{R}_{p-1}^{(p-1)}$, by equation (17). Hence, if $\mathcal{R}_{p-1}^{(p-1)}$ is positive definite, then $\mathcal{R}_p^{(p)}$ is positive definite. But $\mathcal{R}_0^{(0)} = R_0$ is positive definite (see equation (15)). Therefore, we conclude by induction that

$$\mathcal{R}_p^{(p)} \text{ is positive definite for all p.} \tag{31}$$

This statement is used as a priori information in Burg's derivation in the known correlation case (see reference 3, page 85).

Now, we return to equation (30) with this information and can draw the conclusion that $\mathcal{R}_m^{(p)}$ is positive definite for all $m \geq p$. Finally, using equation (17), we can state

$$\mathcal{R}_m^{(p)} \text{ is positive definite for all m and p.} \tag{32}$$

For fixed p, since $\mathcal{R}_m^{(p)}$ is positive definite for all m, (the elements of) $\mathcal{R}_m^{(p)}$ cannot tend to infinity as m tends to infinity, since $R_0^{(p)} = R_0$ is fixed. Therefore, recursion (11) is stable. This implies (using equation (23)[1]) that

$$\det\left(I - \sum_{n=1}^{p} z^{-n} A_n^{(p)}\right) = \det \mathcal{H}_A^{(p)}(z) \tag{33}$$

possesses all its zeros inside the unit circle in the z-plane; that is, predictive error filter $\mathcal{H}_A^{(p)}(z)$ is minimum phase.

The proof above hinges critically on the positive definiteness of U_p, which was demonstrated in the previous section. In particular, this condition is employed in equation (29) to guarantee that the right-hand side be positive.

A word of caution about an apparent alternative proof is worth mentioning here. Having shown that U_p is positive definite, one might be tempted to <u>define</u> $\tilde{R}_m^{(p)}$ by the inverse of equation (165),[1]

$$G^{(p)}(f) = \Delta\, H_A^{(p)}(f)^{-1}\, U_p\, H_A^{(p)}(f)^{-1^H}, \quad |f| < \frac{1}{2\Delta}, \tag{34}$$

according to

$$\tilde{R}_m^{(p)} = \int_{-\frac{1}{2\Delta}}^{\frac{1}{2\Delta}} df\, \exp(i2\pi f m\Delta)\, G^{(p)}(f), \quad \text{all } m. \tag{35}$$

It is obvious that $G^{(p)}(f)$ in equation (34) is positive definite for any f; and it is now easy to demonstrate that $\tilde{\mathcal{R}}_m^{(p)}$ is positive definite:

$$\begin{aligned}
\left[q_0^H \cdots q_m^H\right] \tilde{\mathcal{R}}_m^{(p)} \begin{bmatrix} q_0 \\ \vdots \\ q_m \end{bmatrix} &= \sum_{s,t=0}^{m} q_s^H \tilde{R}_{t-s}^{(p)} q_t \\
&= \sum_{s,t=0}^{m} q_s^H \int_{-\frac{1}{2\Delta}}^{\frac{1}{2\Delta}} df\, \exp(i2\pi f(t-s)\Delta)\, G^{(p)}(f)\, q_t \\
&= \int_{-\frac{1}{2\Delta}}^{\frac{1}{2\Delta}} df \left[\sum_{s=0}^{m} \exp(i2\pi f s\Delta) q_s\right]^H G^{(p)}(f) \left[\sum_{t=0}^{m} \exp(i2\pi f t\Delta) q_t\right] > 0,
\end{aligned} \tag{36}$$

since $G^{(p)}(f)$ is positive definite for any f.

However, the problem is that we now would have to show that $\tilde{R}_m^{(p)}$, as generated by equation (35), satisfies the recurrence (11). An example in appendix C shows that for an unstable sequence, the values returned by equation (35) are not the same sequence; thus, equation (35) should not be used until after the stability of $\{R_m^{(p)}\}$ has been ascertained.

ALIASED CORRELATIONS VIA FFT

Based upon the previous results, we know that we can express

$$G(f) = \Delta \sum_{m=-\infty}^{\infty} \exp(-i2\pi f m\Delta) R_m \, , \quad |f| < \frac{1}{2\Delta} \, , \tag{37}$$

and

$$R_m = \int_{-\frac{1}{2\Delta}}^{\frac{1}{2\Delta}} df \, \exp(i2\pi f m\Delta) \, G(f), \text{ all } m. \tag{38}$$

We have dropped the superscript p above, since the results to follow will hold for any correlation-spectrum pair satisfying equations (37) and (38).

If spectrum G(f) is calculated only at a discrete set of $N_F + 1$ points on $\left(-\frac{1}{2\Delta}, \frac{1}{2\Delta}\right)$ (which is a typical practical situation for plotting purposes, for example), a discrete approximation is afforded to the integral in equation (38). It is, for trapezoidal weights $\{w_k\}$,

$$\frac{1}{N_F \Delta} \sum_{k=-N_F/2}^{N_F/2} w_k \exp\left(i2\pi \frac{k}{N_F\Delta} m\Delta\right) G\left(\frac{k}{N_F\Delta}\right) = \sum_{k=-\infty}^{\infty} R_{m+kN_F} \equiv \hat{R}_m \, . \tag{39}$$

That is, the discrete approximation to integral (38) yields aliased samples of correlation sequence $\{R_m\}$ at separations of N_F; this is easily proven by substituting equation (37) into the left-hand side of equation (39) and interchanging summations.

The aliased sequence $\{\hat{R}_m\}$ has period N_F. Therefore, $\hat{R}_m$ is a good approximation to R_m for $|m| < N_F/2$ if $|R_m|$ is sufficiently small for

$|m| > N_F/2$. (Generally, $N_F \gg p_{BEST}$ in the linear predictive approach, and this is true.) The reason for considering this approach to the approximate evaluation of correlation sequence $\{R_m\}$ follows.

The left-hand side of equation (39) can be accomplished by means of an N_F-point FFT (one FFT for each element of the M × M matrices involved). For trapezoidal weights, using the fact that $G\left(-\frac{1}{2\Delta}\right) = G\left(\frac{1}{2\Delta}\right)$, equation (39) is expressible as

$$\hat{R}_m = \frac{1}{N_F \Delta} \sum_{k=-N_F/2}^{\frac{N_F}{2}-1} \exp(i 2\pi k m / N_F)\, G_k$$

$$= \frac{1}{N_F \Delta}\left[\sum_{k=-N_F/2}^{-1} \exp(i 2\pi k m/N_F)\, G_k + \sum_{k=0}^{\frac{N_F}{2}-1} \exp(i 2\pi k m/N_F)\, G_k\right], \tag{40}$$

where we have defined

$$G_k = G\left(\frac{k}{N_F \Delta}\right), \quad |k| \le \frac{N_F}{2}. \tag{41}$$

Letting $n = N_F + m$ in the first sum of equation (40), and $n = m$ in the second sum, we obtain

$$\hat{R}_m = \sum_{n=0}^{N_F - 1} \exp(i 2\pi n m/N_F)\, \gamma_n , \tag{42}$$

where M × M matrix

$$\gamma_n \equiv \frac{1}{N_F \Delta} \left\{ \begin{array}{ll} G_n , & 0 \le n \le \frac{N_F}{2} - 1 \\ G_{n-N_F} , & \frac{N_F}{2} \le n \le N_F - 1 \end{array} \right\}. \tag{43}$$

But equation (42) is recognized as an N_F-point FFT of the matrices

$$G_0, G_1, \ldots, G_{\frac{N_F}{2}-1}, G_{-\frac{N_F}{2}}, \ldots, G_{-1} ; \tag{44}$$

thus, we obtain $\hat{R}_0, \hat{R}_1, \ldots, \hat{R}_{N_F-1}$ by means of this N_F-point FFT, one FFT for each element of the $M \times M$ matrices. (The quantities $\{\hat{R}_m\}$ for $|m| < N_F/2$ are available by use of the periodic nature of sequence $\{\hat{R}_m\}$.) This use of an N_F-point FFT to obtain (good) estimates of correlation sequence $\{R_m\}$ circumvents the use of recursion (11), which would yield the exact correlation sequence $\{R_m\}$. It can save time in some cases and uses already available quantities $\{G_k\}$, if they have been computed previously for plotting or observation purposes.

REAL PROCESSES

The preceding results for complex multivariate processes can be specialized to real processes. We have, from equations (171)[1] and (39),

$$G_{-k} = G_k^* , \quad \hat{R}_m \text{ real.} \tag{45}$$

Therefore, equation (39) becomes

$$\hat{R}_m = \frac{2}{N_F \Delta} \mathrm{Re} \sum_{k=0}^{N_F/2} \tilde{w}_k \exp(i 2\pi k m / N_F)\, G_k , \tag{46}$$

where

$$\tilde{w}_k \equiv \begin{Bmatrix} \frac{1}{2}, & k=0 \text{ or } N_F/2 \\ 1, & 0 < k < N_F/2 \end{Bmatrix} . \tag{47}$$

Now, let the elements of matrices G_k and $\hat{R}_m$ be expressed as

$$G_k = \left[G_k^{(\ell j)} \right] , \quad \hat{R}_m = \left[\hat{R}_m^{(\ell j)} \right] , \quad 1 \le \ell, j \le M. \tag{48}$$

Then, $G_k^{(\ell\ell)}$ is real for all ℓ; and from equation (46),

$$\hat{R}_m^{(\ell\ell)} = \frac{2}{N_F \Delta} \sum_{k=0}^{N_F/2} \tilde{w}_k \cos(2\pi k m / N_F)\, G_k^{(\ell\ell)} . \tag{49}$$

In addition, since

$$\hat{R}_{-m}^{(\ell\ell)} = \hat{R}_{m}^{(\ell\ell)} , \quad \hat{R}_{\frac{N_F}{2}-m}^{(\ell\ell)} = \hat{R}_{\frac{N_F}{2}+m}^{(\ell\ell)} , \tag{50}$$

the fundamental range of m is $[0, N_F/2]$ for sequence $\{\hat{R}_m^{(\ell\ell)}\}$.

REAL BIVARIATE PROCESSES

We can specialize further to the bivariate case, M = 2, and make use of some of the properties previously discussed. (The goal of these manipulations will not be clear until the final result.) Define the complex scalar sequence $\{u_k\}$ such that

$$u_k = \frac{1}{N_F \Delta} \begin{cases} G_k^{(11)} + i\, G_k^{(22)} , & 0 \le k \le \frac{N_F}{2} - 1 \\ G_{N_F-k}^{(11)} + i\, G_{N_F-k}^{(22)} , & \frac{N_F}{2} \le k \le N_F - 1 \end{cases} . \tag{51}$$

Then,

$$\sum_{k=0}^{N_F-1} u_k \exp\left(\pm i\, 2\pi k m / N_F\right)$$

$$= \frac{1}{N_F \Delta} \sum_{k=0}^{\frac{N_F}{2}-1} \left[G_k^{(11)} + i\, G_k^{(22)}\right] \exp\left(\pm i\, 2\pi k m/N_F\right)$$

$$+ \frac{1}{N_F \Delta} \sum_{k=\frac{N_F}{2}}^{N_F-1} \left[G_{N_F-k}^{(11)} + i\, G_{N_F-k}^{(22)}\right] \exp\left(\pm i\, 2\pi k m/N_F\right). \tag{52}$$

If, on the right-hand side of equation (52), we let n = k in the first sum, and n = N_F - k in the second sum, we get

$$\frac{1}{N_F\Delta}\sum_{n=0}^{\frac{N_F}{2}-1}\left[G_n^{(11)}+i\,G_n^{(22)}\right]\exp\left(\pm i2\pi nm/N_F\right)$$

$$+\frac{1}{N_F\Delta}\sum_{n=1}^{N_F/2}\left[G_n^{(11)}+i\,G_n^{(22)}\right]\exp\left(\mp i2\pi nm/N_F\right)$$

$$=\frac{1}{N_F\Delta}\left\{\left[G_0^{(11)}+i\,G_0^{(22)}\right]+\sum_{n=1}^{\frac{N_F}{2}-1}\left[G_n^{(11)}+i\,G_n^{(22)}\right]2\cos\left(2\pi nm/N_F\right)\right.$$

$$\left.+\left[G_{\frac{N_F}{2}}^{(11)}+i\,G_{\frac{N_F}{2}}^{(22)}\right](-1)^m\right\}$$

$$=\frac{2}{N_F\Delta}\sum_{n=0}^{N_F/2}\hat{w}_n\left[G_n^{(11)}+i\,G_n^{(22)}\right]\cos\left(2\pi nm/N_F\right) \qquad (53)$$

$$=\hat{R}_m^{(11)}+i\,\hat{R}_m^{(22)}\,,$$

the last step by equation (49); that is, using equation (52) again,

$$\left\{\hat{R}_m^{(11)}+i\,\hat{R}_m^{(22)}\right\}_0^{N_F-1}=\mathrm{FFT}_{N_F}\left\{u_k\right\}_0^{N_F-1}\,. \qquad (54)$$

Thus, one N_F-point FFT of scalar sequence $\{u_k\}$, defined in equation (51), will give both (aliased) real scalar autocorrelations $\{\hat{R}_m^{(11)}\}$ and $\{\hat{R}_m^{(22)}\}$; and by the statement under equation (50), $\{\hat{R}_m^{(\ell\ell)}\}$ need be printed out only for $0 \le m \le N_F/2$.

For the crosscorrelation, equation (46) yields

$$\hat{R}_m^{(12)} = \frac{2}{N_F\Delta} \mathrm{Re} \sum_{k=0}^{N_F/2} \tilde{W}_k \exp(i2\pi km/N_F) G_k^{(12)}$$

$$= \frac{2}{N_F\Delta} \mathrm{Re} \sum_{k=0}^{N_F/2} \tilde{W}_k \exp(-i2\pi km/N_F) G_k^{(12)*} \tag{55}$$

$$= \mathrm{Re}\ \mathrm{FFT}_{N_F} \left\{ \tilde{W}_k \frac{2}{N_F\Delta} G_k^{(12)*} \right\}_0^{N_F/2} .$$

This N_F-point FFT of $\frac{N_F}{2} + 1$ nonzero numbers would yield $\left\{\hat{R}_m^{(12)}\right\}_0^{N_F-1}$; and from equation (39), since

$$\hat{R}_{-m} = \hat{R}_m^H \text{ (for general complex } M \times M \text{ matrices)}, \tag{56}$$

it follows (using the periodicity of $\{\hat{R}_m\}$) that for the present case

$$\hat{R}_m^{(21)} = \hat{R}_{-m}^{(12)} = \hat{R}_{N_F-m}^{(12)} . \tag{57}$$

Thus, print out of $\hat{R}_m^{(12)}$ and $\hat{R}_m^{(21)}$ for $0 \leq m \leq \frac{N_F}{2}$ suffices to give complete information about the aliased crosscorrelation. Furthermore, all this information is available from the single N_F-point FFT of equation (55).

In summary, only the two FFTs indicated in equations (54) and (55) need be conducted to obtain complete information about the aliased correlation sequence $\{\hat{R}_m\}$, for $M = 2$. These relations, in addition to the exact correlation recursion (11), have been incorporated in the FORTRAN program listed in appendix D. The comments in appendix K of the earlier report[1] are relevant here also.

SUMMARY

It has been shown above that, for the weighting introduced in equation (136),[1]

$$\Lambda_{p-1} = U_{p-1}^{-1}\ , \quad \Gamma_{p-1} = V_{p-1}^{-1}\ , \quad \text{choice 2}, \tag{58}$$

U_p and V_p are guaranteed positive definite, and the correlation recursion (11) is stable. Therefore, equation (58) is a sufficient condition for the desired properties to hold true. It is not known whether this is a necessary condition, that is, whether equation (58) is the only choice that results in the desired properties of positive definiteness and stability.

However, for M = 1, since, by equation (129),[1] $U_{p-1} = V_{p-1}$, it is possible to show that

$$\Lambda_{p-1} = \Gamma_{p-1} \qquad (M = 1) \tag{59}$$

is the only choice that guarantees the desired properties (see reference 1, page 32). Namely, equations (124),[1] (130),[1] and (114)[1] yield scalar

$$A_p^{(p)} = \frac{\left(\Gamma_{p-1}^{-1} + \Lambda_{p-1}^{-1}\right) Y_N^{(p-1)} Z_{N-1}^{(p-1)*}}{\Gamma_{p-1}^{-1}\left|Z_{N-1}^{(p-1)}\right|^2 + \Lambda_{p-1}^{-1}\left|Y_N^{(p-1)}\right|^2} \quad \text{for } N = p+1,\ M = 1. \tag{60}$$

In addition, if the data samples happen to take on values such that*

$$\left|\frac{Y_N^{(p-1)}}{Z_{N-1}^{(p-1)}}\right| = \left(\frac{\Lambda_{p-1}}{\Gamma_{p-1}}\right)^{1/2}, \tag{61}$$

then

$$\left|A_p^{(p)}\right| = \frac{1}{2}\left[\left(\frac{\Lambda_{p-1}}{\Gamma_{p-1}}\right)^{1/2} + \left(\frac{\Gamma_{p-1}}{\Lambda_{p-1}}\right)^{1/2}\right], \quad (M = 1)\ , \tag{62}$$

---

*If the sample mean of the original data is (made) zero, this choice is not possible for p = 1. For p > 1, the sample means of $\{Y_n^{(p-1)}\}$ and $\{Z_n^{(p-1)}\}$ are not necessarily zero.

which is always larger than 1 (unless $\Lambda_{p-1} = \Gamma_{p-1}$); then, U_p is negative and an unstable correlation recursion results. Thus, equation (59) is the only choice that guarantees positive U_p and a stable correlation recursion, regardless of the data set, for $M = 1$.

It should be noticed that the absolute level of the weights is not specified by equation (59). Thus, for $M \geq 2$, freedom in equation (58), at least to the extent of a common scale factor, must be allowed. Whether this is the only degree of freedom allowed to the choice of Λ_{p-1} and Γ_{p-1} is unknown for $M \geq 2$.

Appendix A

SOME PROPERTIES OF COMPLEX MATRICES

An arbitrary complex square matrix A is called real definite if

$$\mathcal{V}^H A \mathcal{V} = r \text{ (real)} \quad \text{for any } \mathcal{V}, \tag{A-1}$$

where $\mathcal{V}$ is a complex column matrix.

It then follows that

$$A \text{ real definite} \Rightarrow A^H = A, \ \{\lambda_k\} \text{ real}, \tag{A-2}$$

where $\{\lambda_k\}$ are the eigenvalues of A.

For proof, first take the conjugate transpose of equation (A-1),

$$\mathcal{V}^H A^H \mathcal{V} = r \quad \text{for any } \mathcal{V}. \tag{A-3}$$

Subtracting equations (A-1) and (A-3) gives

$$\mathcal{V}^H (A^H - A)\mathcal{V} = 0 \quad \text{for any } \mathcal{V}. \tag{A-4}$$

Therefore,

$$A^H - A = 0, \quad \text{or} \quad A^H = A. \tag{A-5}$$

Also, if $\{V_k\}$ are the eigenvectors of A, then

$$\begin{aligned} A V_k &= \lambda_k V_k, \\ V_k^H A V_k &= \lambda_k V_k^H V_k. \end{aligned} \tag{A-6}$$

Since the left-hand side and $V_k^H V_k$ are real, λ_k is real.

If r in equation (A-1) is positive for any $\mathcal{V} \neq 0$, then A is said to be positive definite. It follows that

$$A \text{ positive definite} \Rightarrow A^H = A, \ \{\lambda_k\} > 0. \tag{A-7}$$

The proof is the same as the proof above, except that now $V_k^H A V_k > 0$ in equation (A-6).

Now, we are in position to prove that

$$\left.\begin{array}{l}\text{A positive definite}\\ \\ \text{B positive definite}\end{array}\right\} \Rightarrow \begin{array}{l}\text{Eigenvalues of AB}\\ \text{are all positive.}\end{array} \qquad \text{(A-8)}$$

For proof, let $\{\lambda_k\}$ and $\{V_k\}$ be the eigenvalues and eigenvectors of AB; then, we have

$$(AB)V_k = \lambda_k V_k$$

$$BV_k = \lambda_k A^{-1} V_k \qquad \text{(A-9)}$$

$$V_k^H B V_k = \lambda_k V_k^H A^{-1} V_k = \lambda_k \left(A^{-1}V_k\right)^H A \left(A^{-1}V_k\right),$$

where we have used $A^H = A$ (equation (A-7)). Since A and B are positive definite, the left-hand side and the factor multiplying λ_k are positive. Therefore, λ_k is positive.

It should be noted that AB need not be Hermitian or positive definite. For example, if

$$A = \begin{bmatrix} \alpha & \beta^* \\ \beta & \alpha \end{bmatrix} \quad \alpha \text{ real}, \alpha > 0, \ \alpha^2 > |\beta|^2,$$

$$B = \begin{bmatrix} \mu & \nu^* \\ \nu & \mu \end{bmatrix} \quad \mu \text{ real}, \ \mu > 0, \ \mu^2 > |\nu|^2, \qquad \text{(A-10)}$$

then,

$$AB = \begin{bmatrix} \alpha\mu + \beta^*\nu & \alpha\nu^* + \mu\beta^* \\ \mu\beta + \alpha\nu & \alpha\mu + \beta\nu^* \end{bmatrix}. \qquad \text{(A-11)}$$

Since the main diagonal terms of AB need not be real, AB is not necessarily Hermitian. Also, if we assume that AB is positive definite, equation (A-7) says that AB is Hermitian, which is contradictory.

A numerical example follows:

$$A = \begin{bmatrix} 2 & 1-i \\ 1+i & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1+i \\ 1-i & 2 \end{bmatrix}. \tag{A-12}$$

A and B are positive definite and Hermitian. The eigenvalues of both are $\{\lambda_k\} = 2 \pm \sqrt{2} > 0$. Their product is

$$AB = \begin{bmatrix} 4-i2 & 4 \\ 4 & 4+i2 \end{bmatrix}, \tag{A-13}$$

with eigenvalues $4 \pm 2\sqrt{3} > 0$, as predicted. But AB is not Hermitian nor positive definite because, for instance,

$$\begin{bmatrix} 1 & 0 \end{bmatrix} AB \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 4 - i2. \tag{A-14}$$

The matrix AB in equation (A-13) points out that specifying a matrix to have positive eigenvalues does not make that matrix positive definite. However, if the matrix is also Hermitian, we have the generalization of equation (A-7) to

$$A \text{ positive definite} \Leftrightarrow A^H = A,\ \{\lambda_k\} > 0. \tag{A-15}$$

Appendix B

RELATIONS OF DETERMINANTS

Since det $Q_m^{(p)} = 1$ (see equation (19)), equation (28) yields

$$\det \mathcal{R}_m^{(p)} = \det U_p \ \det \mathcal{R}_{m-1}^{(p)} \ , \ m \geq p. \qquad \text{(B-1)}$$

Setting m = p in equation (B-1) and employing equation (17), there follows

$$\det \mathcal{R}_p^{(p)} = \det U_p \ \det \mathcal{R}_{p-1}^{(p-1)} \ . \qquad \text{(B-2)}$$

Since $\mathcal{R}_0^{(0)} = R_0 = U_0$ (see equation (95)[1]), this recursion may be written in closed form as

$$\det \mathcal{R}_p^{(p)} = \prod_{k=0}^{p} \det U_k \ . \qquad \text{(B-3)}$$

This relation is given in Burg,[3] page 86.

By letting m = p + 1, p + 2,. . ., in equation (B-1), it follows immediately that

$$\det \mathcal{R}_m^{(p)} = \left(\det U_p\right)^{m-p} \prod_{k=0}^{p} \det U_k \ , \ m \geq p. \qquad \text{(B-4)}$$

In addition, for m < p, using equations (17) and (B-3),

$$\det \mathcal{R}_m^{(p)} = \det \mathcal{R}_m^{(m)} = \prod_{k=0}^{m} \det U_k \ , \ m < p. \qquad \text{(B-5)}$$

Combining equations (B-4) and (B-5), we have

$$\det \mathcal{R}_m^{(p)} = \left\{ \begin{array}{l} \prod_{k=0}^{m} \det U_k \ , \ m \leq p \\ \left(\det U_p\right)^{m-p} \prod_{k=0}^{p} \det U_k \ , \ m \geq p \end{array} \right\} . \qquad \text{(B-6)}$$

Appendix C

EXAMPLE OF UNSTABLE CORRELATION RECURSION

Consider the univariate (M = 1) correlation values,

$$R_m = r^{|m|}, \text{ all } m, \ r \text{ real and positive.} \tag{C-1}$$

The value of r can be greater or less than unity. The z-transform of equation (C-1) is

$$\sum_m z^{-m} R_m = 1 + \sum_{m=1}^{\infty} z^{-m} r^{m} + \sum_{m=-1}^{-\infty} z^{-m} r^{-m} \equiv 1 + S_1 + S_2 . \tag{C-2}$$

Now,

$$S_1 = \frac{r}{z-r} \quad \text{if } |z| > r,$$
$$S_2 = \frac{-z}{z-\frac{1}{r}} \quad \text{if } |z| < \frac{1}{r} . \tag{C-3}$$

But, if $r \geq 1$, there is no common region of convergence; also, sequence $\{R_m\}$ is unstable if $r > 1$. Nevertheless, if we blithely add terms in equation (C-2), we get

$$\sum_m z^{-m} R_m = \frac{\left(r-\frac{1}{r}\right) z}{(z-r)\left(z-\frac{1}{r}\right)} . \tag{C-4}$$

Then, continuing on, setting $z = \exp(i2\pi f\Delta)$ and multiplying by Δ,

$$G(f) \equiv \frac{\Delta\left(r-\frac{1}{r}\right)\exp(i2\pi f\Delta)}{\left[\exp(i2\pi f\Delta)-r\right]\left[\exp(i2\pi f\Delta)-\frac{1}{r}\right]} , \tag{C-5}$$

which is real, and

$$\tilde{R}_m \equiv \int_{-\frac{1}{2\Delta}}^{\frac{1}{2\Delta}} df \exp(i2\pi f m\Delta)G(f) = \frac{1}{i2\pi\Delta}\oint_{\substack{\text{unit}\\ \text{circle}}} \frac{dz}{z} z^m \frac{\Delta\left(r-\frac{1}{r}\right)z}{(z-r)\left(z-\frac{1}{r}\right)} . \tag{C-6}$$

In the following, let $r \neq 1$, $\alpha = \min(r,\frac{1}{r})$, and $\beta = \max(r,\frac{1}{r})$. Then,

$$\tilde{R}_m = \left(r-\frac{1}{r}\right)\frac{\alpha^{|m|}}{\alpha-\beta} \quad \text{for all } m. \tag{C-7}$$

This is a stable sequence for any r. But, notice that if

$$r<1,\ \alpha=r,\ \beta=\frac{1}{r},\ \tilde{R}_m = r^{|m|} \ \text{for all } m; \tag{C-8}$$

whereas, if

$$r>1,\ \alpha=\frac{1}{r},\ \beta=r,\ \tilde{R}_m = -\left(\frac{1}{r}\right)^{|m|} \text{for all } m. \tag{C-9}$$

The former sequence is correct; the latter is not. Yet both are stable. So, although equation (C-6) always generates a stable sequence, it is not necessarily the original sequence.

Appendix D

FORTRAN PROGRAM FOR SPECTRAL ANALYSIS

A FORTRAN listing of the spectral analysis technique is given in this appendix, in addition to a sample printout of an application. The notation and scaling adopted is identical to that given in reference 1, appendix K. The equation numbers referenced are those in the earlier report,[1] except in Subroutine ACM, where they correspond to the equations in this report.

```
C  MULTIVARIATE LINEAR PREDICTIVE SPECTRAL ANALYSIS,
C  EMPLOYING WEIGHTED FORWARD AND BACKWARD AVERAGING.
C  THIS PROGRAM IS WRITTEN FOR REAL PROCESSES AND GENERAL M, WITH THE
C  EXCEPTION OF FUNCTION DETERM AND SUBROUTINES SDM, INVERT, AND SOLVE,
C  AND THE PRINT OUT OF THE SPECTRAL DENSITY MATRIX.
C  USER: CHANGE LINES 24 AND 41, AND REPLACE SUBROUTINE DATA.
C  M = DIMENSIONALITY OF MULTIVARIATE PROCESS; INTEGER INPUT
C  N = NUMBER OF DATA POINTS IN EACH PROCESS; INTEGER INPUT
C  X(1,1)...X(N,1),...,X(1,M)...X(N,M) = INPUT DATA; ALTERED ON OUTPUT
C  PMAX = MAXIMUM ORDER OF FILTER; INTEGER INPUT
C  NF = SIZE OF FFT (MUST BE A POWER OF 2 TO USE MKLFFT); INTEGER INPUT
C  AVE = MEANS OF INPUT DATA; OUTPUT
C  R = COVARIANCE MATRIX OF INPUT DATA; OUTPUT
C  AIC = AKAIKE'S INFORMATION CRITERION; OUTPUT
C  PBEST = BEST ORDER OF FILTER; INTEGER OUTPUT
C  UBEST = MATRIX OF COEFFICIENTS IN SPECTRAL ESTIMATE; OUTPUT
C  AP = MATRIX OF FORWARD PARTIAL CORRELATION COEFFICIENTS; THEN =
C  MATRIX OF FORWARD PREDICTIVE FILTER COEFFICIENTS FOR PBEST; OUTPUT
C  BP = MATRIX OF BACKWARD PARTIAL CORRELATION COEFFICIENTS; OUTPUT
C  RN = MATRIX OF NORMALIZED CORRELATIONS OF INPUT DATA; OUTPUT
C  XX,YY = SPECTRAL MATRICES OF INPUT DATA; OUTPUT
C  XX = ALIASED NORMALIZED CORRELATION MATRIX OF INPUT DATA; OUTPUT
      PARAMETER M=2     @ BIVARIATE PROCESS
      PARAMETER N= 100 , PMAX= 10, NF=1024, NF41=NF/4+1
      INTEGER PBEST,P,LOG2NF,IA
      REAL T,TA,TB
      DOUBLE PRECISION D
      DIMENSION X(N,M),Y(N,M),Z(N,M),UBEST(M,M),AP(M,M,PMAX),
     $BP(M,M,PMAX),AVE(M),XX(NF,M,M),YY(NF,M,M),COSI(NF41),
     $U(M,M),V(M,M),UI(M,M),VI(M,M),A(M,M),B(M,M),R(M,M),RN(M,M,PMAX),
     $WA(M,M),WB(M,M),WC(M,M),WD(M,M),WE(M,M),AIC(PMAX),AICO(2),S(M)
      EQUIVALENCE (X,Y),(AIC(1),AICO(2))
```

```
C  PRINT OUT VALUES OF PARAMETERS
      I=N
      J=PMAX
      K=M
      L=NF
      PRINT 1, I,J,K,L
1     FORMAT(1H1,' N =',I6,10X,'PMAX =',I4,10X,'M =',I2,10X,'NF =',I5)
C  INPUT DATA IN X(1,1)...X(N,1),...,X(1,M)...X(N,M)
      CALL DATA
      PRINT 2
2     FORMAT(/' INPUT DATA:')
      J=N-99
      L=N-200
      DO 3 I=1,M
      PRINT 4, I
      IF(N.LE.200) GO TO 5
      PRINT 6, (X(K,I),K=1,100)
      PRINT 7, L
7     FORMAT(I6,' INPUT DATA POINTS NOT PRINTED HERE')
      PRINT 6, (X(K,I),K=J,N)
      GO TO 3
5     PRINT 6, (X(K,I),K=1,N)
3     CONTINUE
4     FORMAT('   PROCESS NUMBER',I2)
6     FORMAT(5E20.8)
C  EVALUATE PARTIAL CORRELATION COEFFICIENTS
      CALL PCC
      PRINT 8
8     FORMAT(/' MEANS OF INPUT DATA:')
      PRINT 6, (AVE(I),I=1,M)
      PRINT 9
9     FORMAT(/' COVARIANCE MATRIX OF INPUT DATA:')
      PRINT 6, ((R(I,J),I=1,M),J=1,M)
      PRINT 10
10    FORMAT(/' AKAIKE INFORMATION CRITERION:'/
     $9X,'P',11X,'AIC(P)')
      PRINT 11, (P,AIC(P),P=0,PMAX)
```

```
11    FORMAT(I10,E20.8)
      PRINT 12, PBEST
12    FORMAT(/' PBEST =',I3)
      PRINT 13
13    FORMAT(/' UBEST:')
      PRINT 6, ((UBEST(I,J),I=1,M),J=1,M)
      PRINT 14
14    FORMAT(/' FORWARD PARTIAL CORRELATION COEFFICIENTS:'/9X,'P',
     $10X,'A(P,P)11',12X,'A(P,P)21',12X,'A(P,P)12',12X,'A(P,P)22')
      PRINT 15, (P,((AP(I,J,P),I=1,M),J=1,M),P=1,PMAX)
15    FORMAT(I10,4E20.8)
      PRINT 16
16    FORMAT(/' BACKWARD PARTIAL CORRELATION COEFFICIENTS:'/9X,'P',
     $10X,'B(P,P)11',12X,'B(P,P)21',12X,'B(P,P)12',12X,'B(P,P)22')
      PRINT 15, (P,((BP(I,J,P),I=1,M),J=1,M),P=1,PMAX)
      IF(PBEST.EQ.0) GO TO 17
C  EVALUATE PREDICTIVE FILTER COEFFICIENTS
C  AND NORMALIZED CORRELATION MATRICES
      CALL PFC
      PRINT 18
18    FORMAT(/' FORWARD PREDICTIVE FILTER COEFFICIENTS FOR PBEST:'/
     $9X,'K',8X,'A(PBEST,K)11',8X,'A(PBEST,K)21',
     $8X,'A(PBEST,K)12',8X,'A(PBEST,K)22')
      PRINT 15, (P,((AP(I,J,P),I=1,M),J=1,M),P=1,PBEST)
      PRINT 19
19    FORMAT(/' NORMALIZED CORRELATION MATRICES FOR M=2, UP TO PMAX:'/
     $7X,'DELAY',9X,'AUTO11',13X,'CROSS21',13X,'CROSS12',14X,'AUTO22')
      P=0
      PRINT 15, P,((R(I,J),I=1,M),J=1,M)
      PRINT 15, (P,((RN(I,J,P),I=1,M),J=1,M),P=1,PMAX)
C  EVALUATE PREDICTIVE-ERROR FILTER TRANSFER FUNCTION
17    CALL PEFTF
C  EVALUATE SPECTRAL DENSITY MATRIX AND COHERENCE
      CALL SDM
      PRINT 20
```

```
20    FORMAT(/' SPECTRAL DENSITY MATRIX AND COHERENCE FOR M=2:'/
     $8X,'BIN',10X,'AUTO11',14X,'AUTO22',10X,'REAL(CROSS12)',7X,
     $'IMAG(CROSS12)',9X,'MAG SQ COH',11X,'ARGUMENT')
      PRINT 21, (L,XX(L,1,1),XX(L,2,2),XX(L,1,2),
     $YY(L,1,2),YY(L,1,1),YY(L,2,2), L=1,NFD2P1)
21    FORMAT(I10,6E20.8)
C  EVALUATE ALIASED NORMALIZED CORRELATION MATRICES VIA FFT
      CALL ACM
      PRINT 22
22    FORMAT(/' ALIASED NORMALIZED CORRELATION MATRICES FOR M=2:'/
     $7X,'DELAY',9X,'AUTO11',13X,'CROSS21',13X,'CROSS12',14X,'AUTO22')
      L=0
      PRINT 15, L,XX(NFD2P2,1,1),XX(1,2,1),XX(1,2,1),XX(NFD2P2,2,2)
      PRINT 15, (L,XX(NFD2P2+L,1,1),XX(NFP1-L,2,1),
     $XX(1+L,2,1),XX(NFD2P2+L,2,2), L=1,NFD2M2)
      PRINT 15, NFD2M1,XX11M1,XX(NFD2P2,2,1),XX(NFD2,2,1),XX22M1
      PRINT 15, NFD2,XX11M0,XX(NFD2P1,2,1),XX(NFD2P1,2,1),XX22M0
C
      SUBROUTINE DATA
C  THIS SUBROUTINE GENERATES DATA FOR M=2, BIVARIATE PROCESS
      DEFINE IRAND=I*5**15+((1-SIGN(1,I*5**15))/2)*34359738367
      DEFINE RAND=FLOAT(I)/34359738367.
      I=5281
      TA=0.
      TB=0.
      DO 1 K=1,100    @ WILL DISCARD THESE INITIAL POINTS
      I=IRAND
      T=.85*TA-.75*TB+RAND-.5
      I=IRAND
      TB=.65*TA+.55*TB+RAND-.5
```

```
1     TA=T
      X(1,1)=TA
      X(1,2)=TB
      DO 2 K=2,N
      I=IRAND
      T=.85*TA-.75*TB+RAND-.5
      I=IRAND
      TB=.65*TA+.55*TB+RAND-.5
      TA=T
      X(K,1)=TA
2     X(K,2)=TB
      RETURN
C
      SUBROUTINE PCC
C  THIS SUBROUTINE COMPUTES PBEST, UBEST, AND THE PARTIAL
C  CORRELATION COEFFICIENTS FOR P = 1 TO PMAX; ANY M
      I=N
      J=PMAX
      IA=3.*SQRT(N)/M
      IF(PMAX.GT.IA) PRINT 1, J,I,IA
1     FORMAT(/' PMAX =',I4,' IS TOO LARGE FOR NUMBER OF DATA POINTS N ='
     $,I5,'; SEARCH LIMITED TO P =',I4)
      IA=MIN(IA,PMAX)      @ UPPER BOUND ON PMAX; EQ 183
      FAC=2.*M*M/N         @ FAC=0.  WOULD FORCE PBEST EQUAL TO PMAX
C  SUBTRACT MEANS; FILL IN DATA ARRAYS; EQ 11^
      DO 2 I=1,M
      TA=0.
      DO 3 K=1,N
3     TA=TA+Y(K,I)
      TA=TA/N
      AVE(I)=TA
      DO 2 K=1,N
      Y(K,I)=Y(K,I)-TA
2     Z(K,I)=Y(K,I)
```

```
C  INITIALIZE CORRELATION MATRICES; EQS 82, 114, AND 105
      CALL AUTO(2,N-1,Y,WC)
      DO 4 I=1,M
      DO 4 J=I,M
      TA=Y(1,I)*Y(1,J)
      TB=Y(N,I)*Y(N,J)
      R(I,J)=(WC(I,J)+TA+TB)/N
      WA(I,J)=WC(I,J)+TB
      WB(I,J)=WC(I,J)+TA
      R(J,I)=R(I,J)
      WA(J,I)=WA(I,J)
4     WB(J,I)=WB(I,J)
      CALL EQUAL(R,U)
      CALL EQUAL(R,V)
      CALL CROSS(2,N,Y,Y,WC)
C  BEGIN RECURSION
      AIC(0)=LOG(DETERM(U))
      AICMIN=AIC(0)
      PBEST=0
      CALL EQUAL(U,UBEST)
      DO 5 P=1,IA
C  EVALUATE MATRICES REQUIRED IN BILINEAR MATRIX EQUATION; EQ 126
      CALL INVERT(V,VI)
      CALL MULT(VI,WB,WD)
      CALL EQUAL(WD,WB)
      CALL INVERT(U,UI)
      CALL EQUAL(WA,WD)
      CALL MULT(WD,UI,WA)
      CALL ADD(WC,WC,WC)
C  SOLVE BILINEAR MATRIX EQUATION; EQS 157-161
      CALL SOLVE
C  EVALUATE PARTIAL CORRELATION COEFFICIENTS; EQ 124
      CALL MULT(WC,VI,A)
      CALL TRANS(WC,WD)
      CALL MULT(WD,UI,B)
      CALL EQUAL(A,AP(1,1,P))
      CALL EQUAL(B,BP(1,1,P))
```

```
C  UPDATE MATRICES U AND V; EQ 181
      CALL MULT(A,WD,WE)
      CALL SUB(U,WE,U)
      CALL MULT(B,WC,WE)
      CALL SUB(V,WE,V)
C  CALCULATE AKAIKE'S INFORMATION CRITERION; EQ 180
      AIC(P)=LOG(DETERM(U))+FAC*P
      IF(AIC(P).GE.AICMIN) GO TO 6
      AICMIN=AIC(P)
      PBEST=P
      CALL EQUAL(U,UBEST)
6     IF(P.EQ.IA) GO TO 5
C  UPDATE DATA SEQUENCES Y AND Z; EQ 111
      L=P+1
      DO 7 K=N,L,-1
      DO 8 I=1,M
      TA=Z(K-1,I)
      DO 9 J=1,M
9     TA=TA-B(I,J)*Y(K,J)
8     Z(K,I)=TA
      DO 10 I=1,M
      TA=Y(K,I)
      DO 11 J=1,M
11    TA=TA-A(I,J)*Z(K-1,J)
10    Y(K,I)=TA
7     CONTINUE
C  CALCULATE NEW CORRELATION MATRICES; EQ 114
      CALL AUTO(P+2,N,Y,WA)
      CALL AUTO(P+1,N-1,Z,WB)
      CALL CROSS(P+2,N,Y,Z,WC)
5     CONTINUE
      IF(M.EQ.1) RETURN
      K=M-1
      DO 12 I=1,K
      L=I+1
      DO 12 J=L,M
      UBEST(I,J)=.5*(UBEST(I,J)+UBEST(J,I))
```

```
12    UBEST(J,I)=UBEST(I,J)
      RETURN
C
      SUBROUTINE PFC
C  THIS SUBROUTINE COMPUTES THE PREDICTIVE
C  FILTER COEFFICIENTS; ANY M; EQ 79
C  IT ALSO COMPUTES THE NORMALIZED CORRELATION
C  MATRICES, UP TO PMAX; EQS 80A, 25, AND 164
      CALL MULT(AP,R,RN)
      IF(PBEST.EQ.1) GO TO 3
      DO 1 P=2,PBEST
      CALL MULT(AP(1,1,P),R,WC)
      IA=P-1
      DO 2 L=1,IA
      IB=P-L
      CALL MULT(AP(1,1,P),BP(1,1,IB),WA)
      CALL SUB(AP(1,1,L),WA,WA)
      CALL MULT(BP(1,1,P),AP(1,1,L),WB)
      CALL SUB(BP(1,1,IB),WB,BP(1,1,IB))
      CALL EQUAL(WA,AP(1,1,L))
      CALL MULT(WA,RN(1,1,IB),WD)
2     CALL ADD(WC,WD,WC)
1     CALL EQUAL(WC,RN(1,1,P))
3     CONTINUE
      IF(PBEST.EQ.PMAX) GO TO 6
      IA=PBEST+1
      DO 7 P=IA,PMAX
      CALL SUB(WA,WA,WA)
      DO 8 L=1,PBEST
      CALL MULT(AP(1,1,L),RN(1,1,P-L),WB)
8     CALL ADD(WA,WB,WA)
7     CALL EQUAL(WA,RN(1,1,P))
6     DO 4 IA=1,M
```

```
4     S(IA)=1./SQRT(R(IA,IA))
      DO 5 IA=1,M
      DO 5 IB=1,M
      T=S(IA)*S(IB)
      R(IA,IB)=R(IA,IB)*T
      IF(IA.EQ.IB) R(IA,IB)=1.
      DO 5 P=1,PMAX
5     RN(IA,IB,P)=RN(IA,IB,P)*T
      RETURN
C
      SUBROUTINE PEFTF
C  THIS SUBROUTINE COMPUTES THE PREDICTIVE-ERROR
C  FILTER TRANSFER FUNCTION; ANY M; EQS 68 AND (J-3)-(J-6)
      LOG2NF=1.4427*LOG(NF)+.5
      CALL QTRCOS(COSI,NF)
      DO 1 I=1,M
      DO 1 J=1,M
      XX(1,I,J)=0.
      IF(I.EQ.J) XX(1,I,J)=1.
      YY(1,I,J)=0.
      IF(PBEST.EQ.0) GO TO 2
      IA=PBEST+1
      DO 3 L=2,IA
      XX(L,I,J)=-AP(I,J,L-1)
3     YY(L,I,J)=0.
2     IA=PBEST+2
      DO 4 L=IA,NF
      XX(L,I,J)=0.
4     YY(L,I,J)=0.
1     CALL MKLFFT(XX(1,I,J),YY(1,I,J),COSI,LOG2NF,-1)
      RETURN
C
```

```
      SUBROUTINE SDM
C  THIS SUBROUTINE COMPUTES THE SPECTRAL DENSITY
C  MATRIX AND COHERENCE FOR M=2; EQS 178 AND K-5
      T=2./NF
      NFD2P1=NF/2+1
      DO 1 L=1,NFD2P1
      WA(1,1)=XX(L,2,2)
      WA(1,2)=-XX(L,1,2)
      WA(2,1)=-XX(L,2,1)
      WA(2,2)=XX(L,1,1)
      WB(1,1)=YY(L,2,2)
      WB(1,2)=-YY(L,1,2)
      WB(2,1)=-YY(L,2,1)
      WB(2,2)=YY(L,1,1)
      TA=DETERM(WA)-DETERM(WB)
      TB=WA(1,1)*WB(2,2)+WA(2,2)*WB(1,1)-WA(1,2)*WB(2,1)-WA(2,1)*WB(1,2)
      TA=T/(TA**2+TB**2)
      CALL TRANS(WA,WC)
      CALL MULT(UBEST,WC,WD)
      CALL MULT(WB,WD,WC)
      TB=WC(1,2)-WC(2,1)
      CALL MULT(WA,WD,WC)
      CALL TRANS(WB,WD)
      CALL MULT(UBEST,WD,WE)
      CALL MULT(WB,WE,WD)
      CALL ADD(WC,WD,WC)
      YY(L,1,1)=(WC(1,2)**2+TB**2)/(WC(1,1)*WC(2,2))    @ MAG SQ COH
      YY(L,2,2)=ATAN2(TB,WC(1,2))                       @ ARGUMENT
      XX(L,1,1)=TA*WC(1,1)                              @ AUTO11
      XX(L,2,2)=TA*WC(2,2)                              @ AUTO22
      XX(L,1,2)=TA*WC(1,2)                              @ REAL(CROSS12)
      YY(L,1,2)=TA*TB                                   @ IMAG(CROSS12)
      XX(L,2,1)=0.
      YY(L,2,1)=0.
    1 CONTINUE
      RETURN
C
```

```
      SUBROUTINE ACM
C  THIS SUBROUTINE COMPUTES THE ALIASED NORMALIZED CORRELATION
C  MATRICES VIA TWO FFTS, FOR M=2; TECH RPT 5729, EQS 54-57
      NFP1=NF+1
      NFD2=NF/2
      NFD2P1=NFD2+1
      NFD2P2=NFD2+2
      NFD2M1=NFD2-1
      NFD2M2=NFD2-2
C  COMPUTE AUTO CORRELATIONS
      DO 1 L=1,NFD2
      XX(L,2,1)=.5*XX(L,1,1)
      YY(L,2,1)=.5*XX(L,2,2)
      XX(NFD2+L,2,1)=.5*XX(NFD2P2-L,1,1)
1     YY(NFD2+L,2,1)=.5*XX(NFD2P2-L,2,2)
      CALL MKLFFT(XX(1,2,1),YY(1,2,1),COSI,LOG2NF,-1)
C  NORMALIZE AND STORE IN SECOND HALF OF AUTO ARRAYS
      TA=1./XX(1,2,1)
      TB=1./YY(1,2,1)
      T=SQRT(TA*TB)
      XX(NFD2P2,1,1)=1.
      XX(NFD2P2,2,2)=1.
      DO 2 L=2,NFD2M1
      XX(NFD2P1+L,1,1)=XX(L,2,1)*TA
2     XX(NFD2P1+L,2,2)=YY(L,2,1)*TB
      XX11M1=XX(NFD2,2,1)*TA
      XX22M1=YY(NFD2,2,1)*TB
      XX11M0=XX(NFD2P1,2,1)*TA
      XX22M0=YY(NFD2P1,2,1)*TB
C  COMPUTE NORMALIZED CROSS CORRELATIONS
      XX(1,2,1)= .5*XX(1,1,2)*T
      YY(1,2,1)=-.5*YY(1,1,2)*T
      DO 3 L=2,NFD2
      XX(L,2,1)= XX(L,1,2)*T
      YY(L,2,1)=-YY(L,1,2)*T
      XX(NFD2+L,2,1)=0.
```

```
3     YY(NFD2+L,2,1)=0.
      XX(NFD2P1,2,1)= .5*XX(NFD2P1,1,2)*T
      YY(NFD2P1,2,1)=-.5*YY(NFD2P1,1,2)*T
      CALL MKLFFT(XX(1,2,1),YY(1,2,1),COSI,LOG2NF,-1)
      RETURN
C
      SUBROUTINE CROSS(N1,N2,A,B,C)    @ A,B,A NG
C  THIS SUBROUTINE COMPUTES A CROSS CORRELATION MATRIX; ANY M; EQ 114B
      DIMENSION A(N,M),B(N,M),C(M,M)
      DO 1 I=1,M
      DO 1 J=1,M
      D=0.D0
      DO 2 K=N1,N2
2     D=D+A(K,I)*B(K-1,J)
1     C(I,J)=D
      RETURN
C
      SUBROUTINE AUTO(N1,N2,A,B)   @  A,A NG
C  THIS SUBROUTINE COMPUTES AN AUTO CORRELATION MATRIX; ANY M; EQ 114A
      DIMENSION A(N,M),B(M,M)
      DO 1 I=1,M
      DO 1 J=I,M
      D=0.D0
      DO 2 K=N1,N2
2     D=D+A(K,I)*A(K,J)
      B(I,J)=D
1     B(J,I)=B(I,J)
      RETURN
C
      SUBROUTINE EQUAL(A,B)
C  THIS SUBROUTINE SETS TWO MXM MATRICES EQUAL
      DIMENSION A(M,M),B(M,M)
      DO 1 I=1,M
      DO 1 J=1,M
1     B(I,J)=A(I,J)
      RETURN
```

```
C
      SUBROUTINE TRANS(A,B)       @  A,A NG
C   THIS SUBROUTINE TRANSPOSES AN MXM MATRIX
      DIMENSION A(M,M),B(M,M)
      DO 1 I=1,M
      DO 1 J=1,M
1     B(I,J)=A(J,I)
      RETURN
C
      SUBROUTINE ADD(A,B,C)       @  A,B,A OK
C   THIS SUBROUTINE ADDS TWO MXM MATRICES
      DIMENSION A(M,M),B(M,M),C(M,M)
      DO 1 I=1,M
      DO 1 J=1,M
1     C(I,J)=A(I,J)+B(I,J)
      RETURN
C
      SUBROUTINE SUB(A,B,C)       @  A,B,A OK
C   THIS SUBROUTINE SUBTRACTS TWO MXM MATRICES
      DIMENSION A(M,M),B(M,M),C(M,M)
      DO 1 I=1,M
      DO 1 J=1,M
1     C(I,J)=A(I,J)-B(I,J)
      RETURN
C
      SUBROUTINE MULT(A,B,C)      @  A,B,A NG
C   THIS SUBROUTINE MULTIPLIES TWO MXM MATRICES
      DIMENSION A(M,M),B(M,M),C(M,M)
      REAL T
      DO 1 I=1,M
      DO 1 J=1,M
      T=0.
      DO 2 K=1,M
2     T=T+A(I,K)*B(K,J)
1     C(I,J)=T
      RETURN
```

```
C
      SUBROUTINE INVERT(A,B)     @  A,A NG
C  THIS SUBROUTINE INVERTS A 2X2 MATRIX
      DIMENSION A(2,2),B(2,2)
      REAL T
      T=1./DETERM(A)
      B(1,1)=A(2,2)*T
      B(2,2)=A(1,1)*T
      B(1,2)=-A(1,2)*T
      B(2,1)=-A(2,1)*T
      RETURN
C
      SUBROUTINE SOLVE
C  THIS SUBROUTINE SOLVES BILINEAR MATRIX EQUATION
C  FOR M=2, BIVARIATE PROCESS; EQS 157, 158, AND 162
      TA=WA(1,1)+WA(2,2)+WB(1,1)+WB(2,2)
      TB=DETERM(WA)-DETERM(WB)
      CALL MULT(WC,WB,WD)
      WE(1,1)=WA(2,2)
      WE(1,2)=-WA(1,2)
      WE(2,1)=-WA(2,1)
      WE(2,2)=WA(1,1)
      CALL MULT(WE,WC,WA)
      CALL ADD(WD,WA,WD)
      WB(1,1)=TA*WB(1,1)+TB
      WB(2,2)=TA*WB(2,2)+TB
      WB(1,2)=TA*WB(1,2)
      WB(2,1)=TA*WB(2,1)
      CALL INVERT(WB,WE)
      CALL MULT(WD,WE,WC)
      RETURN
C
```

```
      FUNCTION DETERM(A)
C  THIS FUNCTION COMPUTES THE DETERMINANT OF A 2X2 MATRIX
      DIMENSION A(2,2)
      DETERM=A(1,1)*A(2,2)-A(1,2)*A(2,1)
      RETURN
      END

      SUBROUTINE MKLFFT(X,Y,CC,M,ISN)
      DIMENSION X(1),Y(1),CC(1),L(12)
      EQUIVALENCE (L12,L(1)),(L11,L(2)),(L10,L(3)),(L9,L(4)),(L8,L(5)),
     1(L7,L(6)),(L6,L(7)),(L5,L(8)),(L4,L(9)),(L3,L(10)),(L2,L(11)),
     2(L1,L(12))
      N=2**M
      ND4=N/4
      ND4P1=ND4+1
      ND4P2=ND4P1+1
      ND2P2=ND4+ND4P2
      DO 8 LO=1,M
      LMX=2**(M-LO)
      LIX=2*LMX
      ISCL=N/LIX
      DO 8 LM=1,LMX
      IARG=(LM-1)*ISCL+1
      IF(IARG.LE.ND4P1) GO TO 4
      C=-CC(ND2P2-IARG)
      S=ISN*CC(IARG-ND4)
      GO TO 6
    4 C=CC(IARG)
      S=ISN*CC(ND4P2-IARG)
    6 DO 8 LI=LIX,N,LIX
      J1=LI-LIX+LM
      J2=J1+LMX
      T1=X(J1)-X(J2)
      T2=Y(J1)-Y(J2)
      X(J1)=X(J1)+X(J2)
```

```
      Y(J1)=Y(J1)+Y(J2)
      X(J2)=C*T1-S*T2
      Y(J2)=C*T2+S*T1
    8 CONTINUE
      DO 40 J=1,12
      L(J)=1
      IF(J-M) 31,31,40
   31 L(J)=2**(M+1-J)
   40 CONTINUE
      JN=1
      DO 60 J1=1,L1
      DO 60 J2=J1,L2,L1
      DO 60 J3=J2,L3,L2
      DO 60 J4=J3,L4,L3
      DO 60 J5=J4,L5,L4
      DO 60 J6=J5,L6,L5
      DO 60 J7=J6,L7,L6
      DO 60 J8=J7,L8,L7
      DO 60 J9=J8,L9,L8
      DO 60 J10=J9,L10,L9
      DO 60 J11=J10,L11,L10
      DO 60 JR=J11,L12,L11
      IF(JN-JR) 51,51,52
   51 R=X(JN)
      X(JN)=X(JR)
      X(JR)=R
      FI=Y(JN)
      Y(JN)=Y(JR)
      Y(JR)=FI
   52 JN=JN+1
   60 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE QTRCOS(C,N)
      DIMENSION C(1)
      N41=N/4+1
      SCL=6.283185307/N
      DO 1 I=1,N41
 1    C(I)=COS((I-1)*SCL)
      RETURN
      END
```

N = 100 PMAX = 10 M = 2 NF = 1024

INPUT DATA:

PROCESS NUMBER 1

.53901729+00	.24572077+02	-.48246256+00	-.99235174+00	-.62436315+00
.31143945+00	.74571225+02	.94372392+00	.15172541+00	-.76859383+00
-.13781174+00	.25830023+00	.79770139+00	.71260673+00	.80128585+00
.43642798+00	-.44703931-01	-.71200128+00	-.11368756+01	-.92819643+00
-.33429637+00	.57812569+00	.13680694+01	.10164690+01	.50713280+00
-.10550485+01	-.20570143+01	-.19485568+01	-.14135770+01	-.19008934+00
.18380345+01	.31438660+01	.21384055+01	-.46009872+00	-.23113293+01
-.30081896+01	-.28417680+01	-.23547348+00	.16677773+01	.33429194+01
.28077570+01	.56880448+00	-.22631766+01	-.33943964+01	-.26797284+01
-.43547171+00	.21718071+01	.35407598+01	.23288262+01	-.22656890+00
-.25370441+01	-.32986798+01	-.27345279+01	-.40607340+00	.25071375+01
.44460779+01	.32127956+01	.91031587+00	-.13406594+01	-.27867314+01
-.25558064+01	-.11135026+01	.95231916+00	.31850186+01	.37061642+01
.17590479+01	-.89170616+00	-.24854088+01	-.27824243+01	-.13415888+01
.62164557+00	.30034332+01	.35482952+01	.26236263+01	-.89912541-01
-.34339869+01	-.47749634+01	-.34963896+01	-.69929214+00	.24394748+01
.40683676+01	.36433851+01	.54673234+00	-.18884842+01	-.32592577+01
-.20174284+01	-.13084519+01	.12478004+01	.28549910+01	.27696476+01
.13411797+01	-.16488335+01	-.35990672+01	-.33495106+01	-.18694842+01
.17970616+00	.23936473+01	.25152508+01	.16395330+01	-.34817300+00

PROCESS NUMBER 2

.84305659+00	.86934820+00	.16071017+00	.14386062+00	-.81120574+00
-.96936302+00	-.68125464-01	.71946396+00	.53986827+00	-.30419506-01
-.65955505+00	-.32430927+00	.10317814+00	.34722114+00	.80690885+00
.57530412+00	.39259374+00	.14101289+00	.84664986-01	-.19283811+00
-.11810866+01	-.86226903+00	-.20097572+00	.66497141+00	.14060706+01
.11701530+01	.81481338-01	-.81963249+00	-.17306412+01	-.21933125+01
-.17339195+01	.58163615+00	.24263104+01	.29041306+01	.17109564+01
-.18242536+00	-.25142254+01	-.28079124+01	-.19512028+01	.50994501+00

.25400578+01	.30085762+01	.18208549+01	-.84529851+00	-.28813978+01
-.31391280+01	-.17331168+01	.92467879+00	.25048586+01	.30233470+01
.19451272+01	-.49972244+00	-.29138998+01	-.33593366+01	-.25352063+01
.43625407+00	.27920130+01	.31353639+01	.25105703+01	.87293217+00
-.12635761+01	-.25076586+01	-.25748112+01	-.99730413+00	.13507358+01
.26830553+01	.23373865+01	.82668657+00	-.11139500+01	-.21637863+01
-.25236639+01	-.10761964+01	.98666745+00	.30014957+01	.38264174+01
.23522349+01	-.81600191+00	-.32755376+01	-.45958461+01	-.25775154+01
-.21695836+00	.29419152+01	.36481336+01	.28266277+01	-.60319692-01
-.19925641+01	-.27615544+01	-.26589665+01	-.64737901+00	.17339140+01
.31629709+01	.26245975+01	.38355845+00	-.19680776+01	-.27634547+01
-.24165064+01	-.10100275+01	.11536021+01	.19672252+01	.16684788+01

MEANS OF INPUT DATA:

.12068474-01	.29368686-01

COVARIANCE MATRIX OF INPUT DATA:

.46210387+01	.91577268+00	.91577268+00	.37956951+01

AKAIKE INFORMATION CRITERION:

P	AIC(P)
0	.28156752+01
1	-.47116107+01
2	-.46664973+01
3	-.46229553+01
4	-.46316842+01
5	-.45650773+01
6	-.45422159+01
7	-.45357835+01
8	-.44974548+01
9	-.44907100+01
10	-.44389324+01

PBEST = 1

UBEST:

.89002132-01	-.79014897-03	-.79014897-03	.93252867-01

FORWARD PARTIAL CORRELATION COEFFICIENTS:

P	A(P,P)11	A(P,P)21	A(P,P)12	A(P,P)22
1	.87150893+00	.63431677+00	-.77024333+00	.56034775+00
2	-.24060294-01	.15673790+00	-.46089578-01	-.13923085+00
3	.12915299+00	.10394043+00	-.69120812-01	-.12937856+00
4	-.11394597+00	.22871864+00	.18147939+00	.72523244-02
5	.39750100-01	.27629165-01	-.10585018+00	.16504833-01

6	.24732786+00	-.11105661-01	-.13494816-01	-.36488586-02
7	.18216120+00	.10071856+00	.96155114-01	-.10703941+00
8	.19121646+00	.93581546-01	-.25144000-01	.10180847-01
9	.82219766-01	.12731163+00	-.10475666+00	-.22288333+00
10	-.53422038-01	.16761631+00	-.42075946-01	-.26114821-01

BACKWARD PARTIAL CORRELATION COEFFICIENTS:

P	B(P,P)11	B(P,P)21	B(P,P)12	B(P,P)22
1	.56612993+00	-.63141833+00	.77098253+00	.86572675+00
2	-.38369942-01	-.66131392-01	.95437832-01	-.11108892+00
3	.97749089-01	-.38617501-01	.51535051-01	-.11780613+00
4	-.33532630-01	.18097538+00	.21567873+00	.98908506-01
5	-.22097769-02	-.11387029+00	.31994266-01	.28316177-01
6	.22834081+00	.7762 009-01	-.75039910-02	-.71869099-02
7	.20997133+00	.1925 909+00	.53676512-01	-.89760510-01
8	.17425943+00	.3845 344-01	.85870548-01	.43625234-01
9	.39914736-01	-.1197 495+00	.33887566-01	-.22667238+00
10	-.58663233-01	-.7421 017-01	.13284003+00	.24026474-01

FORWARD PREDICTIVE FILTER COEFFICIENTS FOR PBEST:

K	A(PBEST,K)11	A(PBEST,K)21	A(PBEST,K)12	A(PBEST,K)22
1	.87150893+00	.63431677+00	-.77024333+00	.56034775+00

NORMALIZED CORRELATION MATRICES FOR M=2, UP TO PMAX:

DELAY	AUTO11	CROSS21	CROSS12	AUTO22
0	.10000000+01	.2186 251+00	.21864251+00	.10000000+01
1	.71889265+00	.8224 727+00	-.50746841+00	.71338686+00
2	.52424931-01	.9640 763+00	-.94021966+00	.44541466-01
3	-.62724015+00	.5769 243+00	-.85250053+00	-.63314941+00
4	-.94933328+00	-.11577169+00	-.29926953+00	-.95009293+00
5	-.74654176+00	-.7293 963+00	.40236578+00	-.74185658+00
6	-.14151216+00	-.93127804+00	.86849369+00	-.13406150+00
7	.52669119+00	-.6208 877+00	.85047725+00	.53278237+00
8	.89239323+00	.2075 851-01	.36930718+00	.89383619+00
9	.76324072+00	.6362 244+00	-.30205847+00	.75935619+00
10	.22104889+00	.8907 968+00	-.79329062+00	.21407713+00

SPECTRAL DENSITY MATRIX AND COHERENCE FOR M=2:

BIN	AUTO11	AUTO22	REAL(CROSS12)	IMAG(CROSS12)	MAG SQ COH	ARGUMENT
1	.48031416-03	.24469207-03	.10474293-03	.00000000	.93347806-01	.00000000
2	.48039190-03	.24474571-03	.10475898-03	.51851717-05	.93569521-01	.49455843-01
3	.48062528-03	.24490670-03	.10480719-03	.10374396-04	.94234384-01	.98664136-01
4	.48101453-03	.24517527-03	.10488758-03	.15571732-04	.95341385-01	.14738464+00
5	.48156022-03	.24555178-03	.10500028-03	.20781259-04	.96888931-01	.19639113+00
6	.48226298-03	.24603673-03	.10514542-03	.26007074-04	.98874767-01	.24247721+00
7	.48312366-03	.24663077-03	.10532316-03	.31253307-04	.10129598+00	.28846080+00
8	.48414344-03	.24733472-03	.10553375-03	.36524137-04	.10414911+00	.33318695+00
9	.48532356-03	.24814953-03	.10577746-03	.41823766-04	.10743000+00	.37652935+00
10	.48666550-03	.24907630-03	.10605455-03	.47156497-04	.11113392+00	.41839078+00

11	.48817101-03	.25011633-03	.10636541-03	.52526663-04	.11525556+00	.45870120+00
12	.48984205-03	.25127100-03	.10671042-03	.57938682-04	.11978902+00	.49741631+00
13	.49168070-03	.25254195-03	.10709001-03	.63397053-04	.12472785+00	.53451476+00
14	.49368939-03	.25393093-03	.10750468-03	.68906375-04	.13006503+00	.56999511+00
15	.49587071-03	.25543986-03	.10795494-03	.74471342-04	.13579307+00	.60387272+00
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500	.45517991-04	.61373956-04	.74853339-05	.17278155-05	.21125130-01	.22685349+00
501	.45500945-04	.61353407-04	.74823757-05	.15942860-05	.20965359-01	.20993277+00
502	.45485271-04	.61334510-04	.74796546-05	.14609053-05	.20818384-01	.19288885+00
503	.45470968-04	.61317262-04	.74771719-05	.13276617-05	.20684208-01	.17573046+00
504	.45458032-04	.61301665-04	.74749273-05	.11945419-05	.20562828-01	.15846658+00
505	.45446462-04	.61287715-04	.74729191-05	.10615343-05	.20454233-01	.14110678+00
506	.45436256-04	.61275410-04	.74711475-05	.92862562-06	.20358420-01	.12366069+00
507	.45427414-04	.61264746-04	.74696130-05	.79580378-06	.20275391-01	.10613846+00
508	.45419934-04	.61255728-04	.74683152-05	.66305607-06	.20205142-01	.88550377-01
509	.45413816-04	.61248350-04	.74672533-05	.53037026-06	.20147667-01	.70907075-01
510	.45409059-04	.61242612-04	.74664271-05	.39773427-06	.20102963-01	.53219391-01
511	.45405661-04	.6123[illegible]514-04	.74658374-05	.26513530-06	.20071036-01	.35498218-01
512	.45403621-04	.6123[illegible]056-04	.74654829-05	.13256086-06	.20051881-01	.17754633-01
513	.45402941-04	.6123[illegible]236-04	.74653662-05	.29158259-21	.20045497-01	.39058042-16

ALIASED NORMALIZED CORRELATION MATRICES FOR M=2:

DELAY	AUTO11	CROSS21	CROSS12	AUTO22
0	.10000000+01	.2186[illegible]252+00	.218642[illegible]2+00	.10000000+01
1	.71389274+00	.8224[illegible]738+00	-.507468[illegible]1+00	.71338704+00
2	.52424890-01	.9640[illegible]794+00	-.94022[illegible]0+00	.44541553-01
3	-.62724051+00	.5769[illegible]270+00	-.8505[illegible]9+00	-.63314968+00
4	-.94933394+00	-.11577175+00	-.29926977+00	-.95009356+00
5	-.74654241+00	-.7293[illegible]019+00	.402365[illegible]+00	-.74185728+00
6	-.14151228+00	-.93123896+00	.860494[illegible]9+00	-.13406173+00
7	.52669183+00	-.6208[illegible]952+00	.850478[illegible]8+00	.53278291+00
8	.89239447+00	.2075[illegible]831-01	.36930775+00	.89383740+00
9	.76324192+00	.6362[illegible]337+00	-.302059[illegible]8+00	.75935744+00
10	.22104927+00	.8907[illegible]124+00	-.793291[illegible]9+00	.21407763+00
11	-.42912048+00	.6538[illegible]996+00	-.840791[illegible]7+00	-.43530803+00
12	-.83038806+00	.6602[illegible]523-01	-.42890443+00	-.83243695+00
13	-.76977769+00	-.5442[illegible]415+00	.20726147+00	-.76666664+00
14	-.29098326+00	-.8437[illegible]757+00	.71577697+00	-.28452705+00
15	.33536998+00	-.67647757+00	.82241094+00	.34157506+00
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500	-.49072786-02	-.24867039-02	.34092646-03	-.49081368-02
501	-.25524314-02	-.4878[illegible]743-02	.376313[illegible]5-02	-.25471501-02

502	.12059344-02	-.4573[illegible]340-02	.510306[illegible]3-02	.12112272-02
503	.42869227-02	-.1746[illegible]129-02	.362060[illegible]2-02	.42914713-02
504	.49954497-02	.2036[illegible]386-02	.147596[illegible]7-03	.49972947-02
505	.29462978-02	.4667[illegible]941-02	-.33993870-02	.29450784-02
506	-.72473902-03	.4747[illegible]435-02	-.506408[illegible]1-02	-.72796860-03
507	-.39697956-02	.2167[illegible]377-02	-.393160[illegible]3-02	-.39933533-02
508	-.50431756-02	-.1576[illegible]553-02	-.630760[illegible]7-03	-.50506222-02
509	-.33166238-02	-.4465[illegible]668-02	.301475[illegible]2-02	-.33192618-02
510	.24177043-03	-.4887[illegible]877-02	.409309[illegible]3-02	.24286025-03
511	.36667247-02	-.2610[illegible]903-02	.421354[illegible]7-02	.36689682-02
512	.50657272-02	.1107[illegible]301-02	.110743[illegible]1-02	.50683548-02

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